# 2-2 Basic Measures of Time and Frequency Standards

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Time and frequency are the most basic physical quantities which support the modern science and technology. They can be measured most accurately among other physical quantities, and various measuring instruments which deal with time and frequency are supplied. In this paper, the basic concepts, accuracy and stability of time and frequency, which are necessary to evaluate and compare the results obtained by using these instruments, are explained.

***Keywords***
**Accuracy, Uncertainty, Stability, Two sample variance (Allan variance), Power spectrum density**

## 1 Introduction

As science and technology continue to progress, high accuracy and high reliability have become primary requirements in every field in which measurement plays a role. And indeed, among the physical quantities, time and frequency can be measured with the highest accuracy and reliability. The time and frequency quantity pair is essential not only to the establishment of time and frequency standards but also to many basic measurement devices such as frequency synthesizers, spectrum analyzers, network analyzers, and synchroscopes. These measurement devices are used at a variety of research sites and manufacturing plants, with a wide range of applications and objectives. Due to the high accuracy and sophisticated functions of these recent measurement devices, it is critical to understand not only their means of operation, performance limits, and error factors, but also the physical significance of the various measurement target quantities. This paper describes the concepts of accuracy and stability that must serve as the basic measures in precise time and frequency measurement.

## 2 Basic measures for time and frequency measurement

### 2.1 Evaluation of frequency accuracy and uncertainty

The ITU-R TF.686-1 GLOSSARY[1]defines accuracy as follows.
Accuracy: The degree of conformity of a measured or calculated value to its definition.

In other words, accuracy represents the degree to which the measurement result agrees with a given definition. The defined value for time and frequency is set forth in the "Definition of Time and Frequency and International Atomic Time/Coordinated Universal Time." The defined value of time and frequency is physically realized in the Cs primary frequency standard, and this value is relayed to subordinate standards under an established framework of traceability. This relay process is naturally accompanied by errors, reducing the reliability of measurement. The reliability of these results is measured in terms of uncertainty. Reference[1]defines uncertainty as follows.
Uncertainty: The limits of the confidence interval of a measured or calculated quantity.

Thus, referring to accuracy only makes

sense when one has assessed uncertainty. The means of evaluating uncertainty are described in detail in the ISO's "Guide to the Expression of Uncertainty in Measurement;"[2]therefore this paper will address the assessment process in brief.

The process of evaluating uncertainty is divided into the following steps:

(1) Development of a mathematical model for measurement

(2) Evaluation of uncertainty components

(3) Calculation of combined standard uncertainty

(4) Determination of expanded uncertainty

In the development of a mathematical model (1), the measurement process is first clarified, the error components are enumerated, and the relation between measurement $Y$ and input $X_i$ is expressed in mathematical form: $Y = f(X_1, X_2, ..., X_N)$. If $f$ is not given by a theorem, the $X$-$Y$ relation must be found by experiment. In the evaluation of the components of uncertainty (2), standard uncertainty $u(x_i)$ is calculated for the estimate of each input, $x_i$. Standard uncertainty $u(x_i)$ of estimate $y$ of $Y$ is given by the following equation as combined standard uncertainty $u_c(y)$ by appropriately combining $u(x_i)$.

$$u_c^{\,2}(y) = \sum_{i=1}^{N} \left[ \frac{\partial f}{\partial x_i} \right]^2 u^2(x_i) \qquad (1)$$

Note that the above equation assumes that there is no correlation among various values of $x_i$. If there is such a correlation, covariants must be considered. In practical measurement, individual measurements of $Y$ are distributed near $y$, and most of the measurement results lie within the range $y \pm ku_c(x_i)$ within a certain confidence level, where $k$ is the coverage factor and $ku_c(x_i)$ is referred to as "expanded uncertainty," and depends on the assumed confidence level. In the case of a normal distribution, $k$ = 2 implies a 95% confidence level and $k$ = 3 implies a 99% confidence level.

Meanwhile, output frequency $\nu$ of the primary frequency standard is expressed by the following equation.

$$\nu = \nu_o + \sum \frac{\partial \nu}{\partial x_i} dx_i \qquad (2)$$

where $\nu_0$ is a defined value, each $x_i$ is a frequency shift variance (related, for example, to magnetic field shift, second order Doppler shift, resonator cavity frequency shift, black body radiation shift, or shift in gravity potential). For the details of individual shift factors, see Reference[3]. Evaluating the accuracy of a primary frequency standard is equivalent to evaluating the frequency shifts of Eq. (2) and evaluating its uncertainty, $u(x_i)$.

Additionally, the frequency $\nu'$ of a lower-rank standard, such as a secondary frequency standard, is compared with that of a higher-rank standard for evaluation.

$$\nu' = \nu + m \qquad (3)$$

Here m is the frequency comparison/measurement result, and its uncertainty is comprised of $u(\nu)$ and $u(m)$, the uncertainties of the higher-rank standard and the measurement, respectively. If a frequency comparison link such as the GPS common-view method is employed, $u(m)$ includes uncertainty attributable to the frequency comparison link, in addition to the uncertainty specific to the employed frequency comparison/measurement system.

## 2.2 Frequency instability

Reference[1]defines frequency instability as follows.

Frequency instability: Spontaneous and/or environmentally induced frequency change within a given time interval.

In other words, frequency instability represents the degree to which the output frequency of a frequency standard remains constant over a given period of time. Since in most time and frequency applications the atomic frequency standard is operated as a clock over a long period of time, in order to assess frequency instability a measure is required that can express instability not only in the short term but also over long periods—from several months to a year or more.

There are two sources of frequency fluctuations: noise and external disturbances affecting the oscillator and measurement system. Noise can be classified into five types, as described later, in accordance with the various mechanisms of noise generation. Among these types of noise, the classical variance of frequency fluctuation resulting from flicker FM noise or random walk FM noise diverges if these fluctuations are averaged over infinite time. Thus classical variance cannot be used as the measure of frequency instability[4]. External disturbances come from sources such as temperature and magnetic fields, and may consist of periodic disturbances or long-term drifts. Frequency fluctuations differ according to what is termed the frequency "fluctuation factor." It may be possible to determine the dominant fluctuation factors based on the actual behavior of frequency fluctuation within the applied system, thereby improving system performance.

There are two indexes of frequency instability: one for frequency domain and the other for time domain. The index for frequency domain represents frequency fluctuations in the form of a power spectrum density indicating the intensity of slow or rapid frequency fluctuation. The index for time domain, in turn, represents temporal frequency fluctuations averaged over $\tau$ seconds. Each of these indexes can be converted into the other through a given conversion law. The index for frequency domain is suited for the expression of rapid frequency changes (the Fourier frequency, for example, at approximately 1 Hz or more, with an averaging time of one second or less), and is often used to express the purity of the signal spectrum affected by additive phase noise. In contrast, because the time domain index is appropriate to express frequency and phase stability over relatively long time periods, it is often used in time and frequency standards field. In principle, the number of measurement points ($N$) should be sufficiently large and the variance with respect to the mean value of these points must be taken into consideration when calculating the index for time domain. However, as described earlier, there is a problem in that frequency fluctuation due to flicker FM noise or random walk FM noise diverges, as $N \to \infty$. An established solution to this problem is to calculate variances for a definite $N$ and to then average them over infinite time, to avoid divergence. This method, in which $N = 2$, is referred to as two-sample variance (or Allan variance) and serves as the basis of stability of the time domain index.

Frequency and phase fluctuations are basically random phenomena, expressed in terms of statistical quantities. These statistical expressions of frequency or phase fluctuation are described below, beginning with the expression of the output signal $V(t)$ of an oscillator.

$$V(t) = [V_0 + \varepsilon(t)]\sin[2\pi\nu_0 t + \phi(t)] \quad (4)$$

Here, $V_0$ and $\nu_0$ are nominal values of the amplitude and frequency of the output signal, respectively, and $\varepsilon$ and $\phi$ express the amplitude fluctuation and phase fluctuation, respectively. If instantaneous phase value $\Phi(t)$ is given by:

$$\Phi(t) = 2\pi\nu_0 t + \phi(t) \quad (5)$$

then instantaneous frequency $\nu(t)$ is given by:

$$\nu(t) = (d\Phi / dt)/2\pi = \nu_0 + \dot{\phi}(t)/2\pi \quad (6)$$

Because the signals dealt with in standard time and frequency applications are very pure, $\varepsilon$ and $\phi$ are therefore generally very small, and the following can be assumed.

$$|\varepsilon(t)/V_0| \ll 1 \quad (7)$$

$$\left|\dot{\phi}(t)/2\pi\nu_0\right| \ll 1 \quad (8)$$

In the description below, the relative value $y(t)$, representing the frequency fluctuation normalized by the nominal value, is used.

$$y(t) = \dot{\phi}(t)/2\pi\nu_0 \quad (9)$$

Additionally, time fluctuation $x(t)$ is defined as follows:

$$x(t) = \phi(t)/2\pi\nu_0 \tag{10}$$

The employed statistical quantity differs depending on whether fluctuations $y(t)$ and $x(t)$ are considered in time or frequency domain. When they are considered along the time axis, the variance or auto-correlation function $R_y(\tau)$ is used as the basic quantity.

$$R_y(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_0^T y(t)y(t+\tau)dt = \langle y(t)y(t+\tau)\rangle \tag{11}$$

Here < > expresses averaging over infinite time. On the other hand, when the fluctuations are considered along the frequency axis, the power spectrum density $S_y(f)$ is taken as the basic quantity. This is calculated by averaging the squares of y($t$, $f$, $\Delta f$) obtained by passing $y(t)$ through a narrow ($f$ to $f+\Delta f$) band-pass filter and converting to a unit bandwidth.

$$S_y(f) = \lim_{\Delta f\to 0}\frac{1}{\Delta f}\langle y(t,f,\Delta f)^2\rangle \tag{12}$$

As is well known, $R_y(\tau)$ and $S_y(f)$ are related by Wiener-Khintchine's equation.

$$S_y(f) = 4\int_0^\infty R_y(\tau)\cos(2\pi f\tau)d\tau \tag{13}$$

$$R_y(\tau) = \int_0^\infty S_y(f)\sin(2\pi f\tau)df \tag{14}$$

Here, $S_y(f)$ is a one-sided spectrum density defined within $0 \le f \le \infty$. Equations (9) and (10) provide the relation between $S_y(f)$ and phase and time fluctuations $S_\phi(f)$ and $S_x(f)$ as follows.

$$S_y(f) = \omega^2 S_x(f) = (f/\nu_0)^2 S_\phi(f) \tag{15}$$

The RF spectrum $S_{RF}(f)$ is obtained by observing the signals of Eq. (4) with a spectrum analyzer. When the requirements of Eqs. (7) and (8) are fulfilled, this value is related with $S_\phi(f)$ and expressed below as a side-band ratio $L(f)$ to carrier wave level $C$.

$$L(f) = S_{RF}(\nu_0 + f)_{PM}/C \cong S_\phi(f)/2 \tag{16}$$

$S_y(f)$ can be expressed by a polynomial of Fourier frequency f as follows.

$$S_y(f) = \sum_{\alpha=-2}^{2} h_\alpha f^\alpha \tag{17}$$

In this equation, $\alpha = 2$ indicates white PM noise, $\alpha = 1$ is flicker PM noise, $\alpha = 0$ is white FM noise, $\alpha = -1$ is flicker FM noise, and $\alpha = -2$ is random walk noise.

White PM noise having the largest $\alpha$ becomes dominant when $f$ is sufficiently large. This is caused by the additive noise that always overlaps signals generated with the oscillator. The additive noise in the low-frequency region of electromagnetic wave (including microwaves) is thermal noise $kT$, while it is quantum noise $h\nu_0$ in the optical frequency region. The form of the power spectrum density is $kTf^2/\nu_0^2$ or $h\nu_0 f^2/\nu_0^2 P$ depending on individual noise properties[5], where $P$ is the output power of the oscillator, $\nu_0$ the oscillator frequency, and $h$ Planck's constant.

Flicker PM noise is produced by phase modulation by flicker noise[4] due to the non-linearity of circuit devices (those constituting amplifiers, for example).

White FM noise is included in both active and passive frequency standards. Because white FM noise is produced by disturbed oscillation due to noise in the oscillator loop within the oscillator (active standards), its amplitude depends on the oscillator Q value, which represents the sharpness of oscillation. If the half-width of the oscillator spectrum is $\Delta\nu$, then Q = $\nu_0/\Delta\nu$. The power spectrum density[5] is given by $kT/PQ^2$ in the microwave standard (masers and the like) and $h\nu_0/PQ^2$ in the optical-frequency standard (lasers and the like). In the microwave standards, since the spectrum width of the atomic transition line determines the loop bandwidth of the oscillator, $Q$ is given as the spectrum $Q$ of the atomic transition line. Meanwhile, in optical frequency standards, since the resonator $Q$ is generally larger than the spectrum $Q$ of the atomic transition line, the resonator $Q$ determines the loop bandwidth. In the passive frequency standard, white FM noise is produced, as the frequency control loop is disturbed by noise intrinsic to the detection of resonance.

Flicker FM noise and random walk noise are factors limiting the long-term stability of all frequency standards. These types of noise are also present in quartz resonators, in which the flicker noise level is proportional to $1/Q^4$, where $Q$ is resonator $Q$. Flicker frequency noise and random walk noise are also produced in electronic circuits and frequency standards, depending on environmental conditions[5].

In actual frequency measurement, instantaneous frequency $y(t)$ is not measured, while frequency $\bar{y}_k$ (frequency averaged over frequency counter gate time or phase measurement averaging time $\tau$), is measured.

$$\bar{y}_k = \frac{1}{\tau}\int_{t_k}^{t_k+\tau} y(t)dt = \left[\phi(t_k+\tau) - \phi(t_k)\right]/2\pi\nu_0$$

$$= \left[x(t_k+\tau) - x(t_k)\right]/\tau \qquad (18)$$

Variance $s^2$ of time series data of $\bar{y}_k$, which is obtained by $N$ instances of measurement of $\tau$-second average frequency at measurement intervals of $T$ seconds, is an index representing the magnitude of fluctuation of $\bar{y}_k$.

$$s^2 = \frac{1}{N-1}\sum_{n=1}^{N}\left(\bar{y}_n - \frac{1}{N}\sum_{k=1}^{N}\bar{y}_k\right) \qquad (19)$$

Although the above equation expresses an often-used classical unbiased variance, the terms for flicker FM noise and random walk noise diverge when averaged over infinite time as $N\rightarrow\infty$, and the magnitude of variance varies with measurement time and sampling methods in measurement over a definite period of time. Thus the above index is not an appropriate index of signal frequency instability. This problem was studied in the 1960s by Allan, Barnes, and others, and the variance expressed by the following equation was defined as a frequency instability index for time domain.

$$\left\langle \sigma_y^2(N,T,\tau)\right\rangle = \left\langle \frac{1}{N-1}\sum_{n=1}^{N}\left(\bar{y}_n - \frac{1}{N}\sum_{k=1}^{N}\bar{y}_k\right)\right\rangle \qquad (20)$$

Although Eq. (19) is similar to Eq. (20), there is a fundamental difference between the two. Equation (19) assumes that the variance averaged over infinite time is provided as $N\rightarrow\infty$, whereas Eq. (20) defines a sample variance for a definite time range $NT$ and avoids the divergence between flicker FM noise and random walk noise by averaging them over infinite time. Particularly when $T = \tau$ at $N = 2$, the following simple equation (referred to as the Allan variance or as two-sample variance) is given.

$$\sigma_y^2(\tau) = \left\langle \left(\bar{y}_{i+1} - \bar{y}_i\right)^2/2\right\rangle \qquad (21)$$

**Table 1** Relationship between the measures for frequency instability

($f_h$: measurement band width)

| | $\sigma_y^2(\tau)$ $[N=2,\ T=\tau]$ | $\left\langle\sigma_y^2(N,\tau,\tau)\right\rangle$ |
|---|---|---|
| $f^0$PM noise $S_y(f)=h_2f^2$ $2\pi f_h\tau >> 1$ | $h_2\frac{3f_h}{(2\pi\tau)^2}$ | $h_2\frac{N+1}{N(2\pi)^2}\cdot\frac{2f_h}{\tau^2}$ |
| $f^{-1}$PM noise $S_y(f)=h_1f$ $2\pi f_h\tau >> 1$ $2\pi f_hT >> 1$ | $h_1\frac{1}{(2\pi\tau)^2}\{3[2+\ln(2\pi f_h\tau)]-\ln 2\}$ | $h_1\frac{2(N+1)}{N(2\pi\tau)^2}\left[2+\ln(2\pi f_h\tau)-\frac{\ln N}{N^2-1}\right]$ |
| $f^0$FM noise $S_y(f)=h_0$ | $h_0\frac{1}{2}\tau^{-1}$ | $h_0\frac{1}{2}\tau^{-1}$ |
| $f^{-1}$FM noise $S_y(f)=h_{-1}f^{-1}$ | $h_{-1}\cdot 2\ln 2$ | $h_{-1}\frac{N\ln N}{N-1}$ |
| $f^{-2}$FM noise $S_y(f)=h_{-2}f^{-2}$ | $h_{-2}\frac{(2\pi)^2\tau}{6}$ | $h_{-2}\frac{(2\pi)^2\tau}{12}N$ |

Equations (21), (20), and (17) have certain mathematical relationships, as shown in Table 1(*6), and each may be substituted in the others. The Allan variance has proven extremely useful and is now widely employed.

The frequency instability of a practical frequency standard is affected by temperature fluctuations, external noise, and external disturbances, in addition to the abovementioned noise. For example, if the temperature of the laboratory changes periodically due to the effects of air conditioning, the oscillator frequency may vary. Now suppose that a hydrogen maser with $\tau^{-1/2}$ property operates in a laboratory room and that room temperature changes at intervals of $T_m$ induces the maser frequency fluctuation at intervals of $T_m$ through cavity pulling effect. As a result, the maser stability presents a pattern in which the $\tau^{-1/2}$ curve features a bump at around $\tau$ of $T_m/2$. If such a pattern is recognized, frequency instability can be improved by reducing room-temperature fluctuations or improving the temperature properties of the maser itself. Meanwhile, a simple calculation shows that $\sigma_y(\tau)$ is proportional to $\tau$ if the output frequency of the oscillator drifts with temperature.

While the Allan variance serves as a very useful, widely adopted index, white PM noise and flicker PM noise both feature $\tau^{-2}$ forms and thus cannot be distinguished from each other. To solve this problem, a modified Allan variance Mod $\sigma_y^2(\tau)$ has been proposed[7]. This Mod $\sigma_y^2(\tau)$ is defined as follows.

$$\mathrm{Mod}\ \sigma_y^2(\tau) = \frac{1}{2}\left\langle \left( \frac{1}{n}\sum_{j=1}^{n}\left(\bar{y}_{j+n} - \bar{y}_j\right)\right)^2 \right\rangle \qquad (22)$$

$$\bar{y}_{j+n} = \frac{1}{\tau}\int_{t_0+(j+n)\tau_0}^{t_0+(j+2n)\tau_0} y(t')dt' \qquad (23a)$$

$$\bar{y}_j = \frac{1}{\tau}\int_{t_0+j\tau_0}^{t_0+(j+n)\tau_0} y(t')dt' \qquad (23b)$$

where $\tau = n\tau_0$ and $\tau_0$ is constant and $n$ is proportional to $\tau$. For white PM noise, Mod $\sigma_y^2(\tau)$ is proportional to $\tau^{-2}n^{-1}$. On the other hand, for flicker PM noise, it shows a $\tau^{-2}$ pattern when $n$ is sufficiently large, allowing these two types of noise to be distinguished from each other.

In addition, indexes referred to as the time interval error (TIE) and the time variance have in recent years begun to be applied. This paper will introduce only definitions of these indexes below; please see Reference[8] for more details. Among other applications, the TIE index is also used in synchronization in digital communications networks, and is defined as follows.

$$\mathrm{TIE}(t,\tau) = x(t+\tau) - x(t) - \tau y(t,\tau) \qquad (24a)$$

$$y(t,\tau) = \frac{1}{\tau}\int_{t-\tau}^{t} y(t')dt' \qquad (24b)$$

TIE indicates the extent of the time difference produced in $\tau$ seconds when two oscillator signals having the same frequency and time at $t$=0 are output. The statistically expected value of TIE is expressed by an Allan variance as follows.

$$\mathrm{E}\left[TIE^2(t,\tau)\right] = 2\tau^2\sigma_y^2(\tau) \qquad (25)$$

Time variance $\sigma_x^2(\tau)$ is used as an index of network performance and is defined as follows:

$$\sigma_x^2(\tau) = \frac{1}{3}\tau^2 \mathrm{Mod}\ \sigma_y^2(\tau) \qquad (26)$$

## 3 Conclusions

The time and frequency physical quantity pair is the most basic element used to describe natural phenomena. In addition, because it can be measured with far higher accuracy than the other remaining physical quantities, time and frequency plays a crucial role in many areas of modern science and technology. Consequently, a large number of high-performance time and frequency measurement devices are now available. This paper has briefly explained the basic concepts, accuracy and frequency instability, which are required to evaluate such measurement devices and the results they produce.

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