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DETERMINATION OF ELEMENTS OF SUB-SYNCHRONOUS OR SUPER-SYNCHRONOUS ORBITS AND THEIR APPLICATION TO OPERATIONAL SATELLITES

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ABSTRACT

First, sub-synchronous and super-synchronous orbits are defined according to the terms adopted by CCIR where the direction of the reference plane defining the orbital period is constant, and the characteristics of these orbits and a method of determining the orbits are shown. Then, the above definitions of the orbits are improved with the result that the direction of the reference plane defining the orbital period rotates with the period of the earth's rotation, and the characteristics of the orbits and a method of determining the orbits are shown. The relation between these sub-synchronous or super-synchronous orbits and recurrent or sun-synchronous orbits is also mentioned. Lastly, the examples of sub-synchronous orbits which seem to be applicable to operational satellites are given.

1. Introduction

According to terms and definitions relating to space radiocommunications adopted by CCIR[1], a synchronous satellite is a satellite for which the mean sidereal period of revolution about the earth is equal to the sidereal period of rotation of the earth about its axis, a sub-synchronous satellite is a satellite for which the period of revolution is a sub-multiple of the period of rotation of the earth, and a super-synchronous satellite is a satellite for which the period of revolution is an integral multiple of the peroid of rotation of the earth. The sidereal period of revolution of a satellite is defined as the time elapsing between two successive passages of a satellite in the same sense through a reference plane including the centre of mass of the earth and pointing in a fixed direction relative to an inertial system. The above definition of the satellite orbits becomes unique only in case of neglecting the orbital perturbation, but becomes vague in case of real perturbed orbits, because of ununiqueness of the sidereal period of revolution caused by arbitrariness of the direction of the reference plane. For example, if the equatorial plane is adopted as the reference plane, the sidereal period of revolution equals the nodal peroid, and no synchronous satellites come to synchronize with the earth's rotation. On the contrary, if a plane perpendicular to the equatorial plane, i.e. the reference plane defining the earth's rotation, is adopted as the reference plane, synchronous satellites synchronize with the earth's rotation, sub-synchronous satellites pass at nearly the same longitude or right ascension at the same mean sidereal time every sidereal day, and super-synchronous satellites pass at the same longitude or right ascension at nearly the same mean sidereal time at intervals of a few sidereal days. No such sub-synchronous or super-synchronous satellites as defined above will be used as operational satellites for observation or communications, except for astronomical observation. But adopting the plane including the earth's rotational angular moment and rotating with the earth's mean revolutional period (this plane nearly equals the plane including the earth's axis and the sun) as the reference plane, we have sub-synchronous satellites passing at nearly the same longitude at the same time every day, and super-synchronous satellites passing at the same longitude at nearly the same time at intervals of a few days. These satellites will be used as operational satellites for meteorological, earth resources or geodesic observations or for broadcasts.

2. A Method of Determining Sub-Synchronous or Super-Synchronous Orbits

2-1 In case of the reference plane pointing a fixed direction

According to terms relating to sub-synchronous, synchronous or super-synchronous orbits adopted by CCIR, these orbits are defined by the use of the reference plane including the earth's axis and pointing a fixed direction relative to an inertial system as follows:

$$T_f = T_\theta / N \qquad (1)$$

where T_f : time elapsing between two successive passages of a satellite in the same sense through the reference plane.

T_θ : the earth's mean rotational period = 0.9997269672 day = 86164.09966 sec (≒ a sidereal day)

N : synchronous number. In case of N being a positive integer larger than one a sub-synchronous orbit, in case of N being equal to one a synchronous orbit, and in case of N being a reciprocal of a positive integer larger than one a super-synchronous orbit.

Orbits are defined as sub-synchronous, synchronous and super-synchronous orbits, depending on value of N being a positive integer larger than 1, being equal to 1 and being a reciprocal of a positive integer larger than 1, respectively. Fig. 1, where OAB is the equatorial plane and NOSAC is the reference plane, shows that T_f is the elapsing time for which a satellite makes one revolution from A to C round the earth, and that during this time the ascending node rotates from A to B. Meanwhile the elapsing time for which a satellite makes one revolution from A to B is shown by the following formula[2].

$$T_n = 2\pi / (n + \dot{\omega}) \qquad (2)$$

where T_n : nodal period

n : mean motion

$\dot{\omega}$: motion of argument of perigee.

This formula shows that the mean revolutional angular speed of a satellite relative to the ascending node equals $n + \dot{\omega}$. In general, the mean period of the projected motion of a

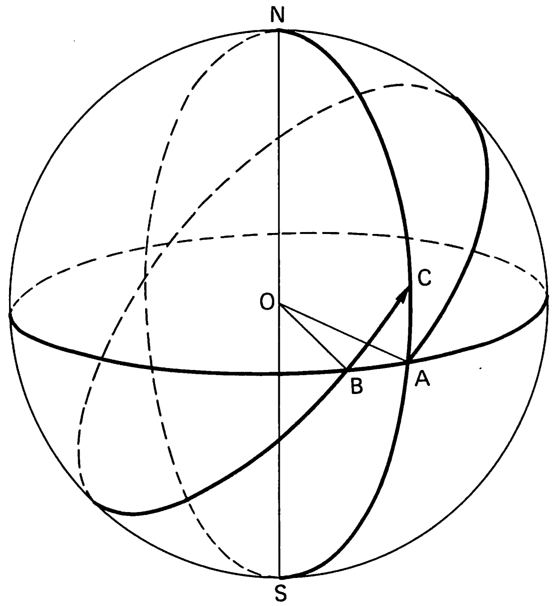

Fig. 1 Relation of a reference plane to define periods (ACNOS), an orbital plane (OABC) and the equatorial plane (OAB).

satellite on the equator equals the mean revolutional period of the satellite. Therefore, the projected motion of a satellite on the equator has the mean angular speed equal to $\pm(n + \dot{\omega})$ relative to the ascending node, whose plus sign is adopted in case of $0 \leqslant i \leqslant \pi/2$ and negative sign is adopted in case of $\pi/2 < i \leqslant \pi$ where i is orbital inclination. But supposing that where $\pi/2 < i \leqslant \pi$ the sign of $(n + \dot{\omega})$ is spontaneously altered in case of projection of satellite motion on the equator, we do not need the above minus sign. This supposition is adopted in References (2) & (3), and also widely used on the occasion of calculating satellite orbits (refer to (13)). Namely, only the upper sign of $(n + \dot{\omega})$ is necessary if the signs of n and $\dot{\omega}$ of retrograde orbits spontaneously change. The mean revolutional angular speed of the projected motion of the satellite on the equatorial plane in an inertial system is $\pm(n + \dot{\omega} + \dot{\Omega}$, where $\dot{\Omega}$ is angular speed of the ascending node in an inertial system. Therefore,

$$T_f = \frac{\pm 2\pi}{\pm(n + \dot{\omega}) + \dot{\Omega}} \tag{3}$$

Hereinafter upper and lower signs correspond to direct and retrograde orbits, respectively.

Expressing the rotational period of the earth in angular speed, we have

$$T_\theta = 2\pi / \mu \tag{4}$$

where μ: mean rotational angular speed of the earth.

Substituting (3) and (4) into (1), we have

$$\pm(n + \dot{\omega}) + \dot{\Omega} = \pm N\mu \tag{5}$$

or

$$n + \dot{\omega} \pm \dot{\Omega} = N\mu \tag{5 }$$

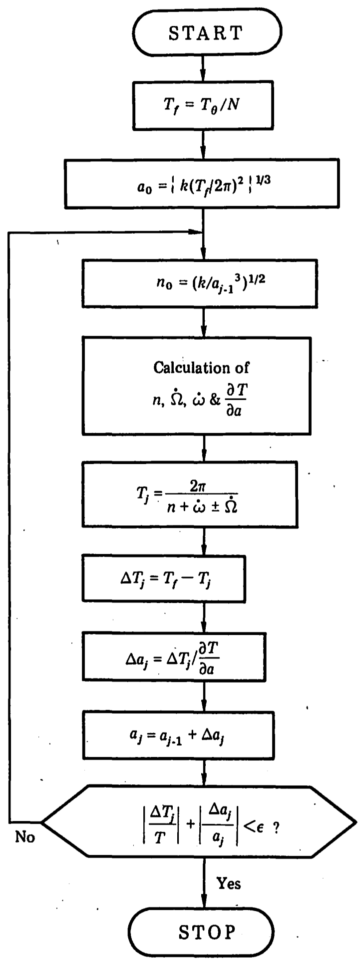

Fig. 2 The flow chart to calculate a semi-major axis a of a sub-synchronous, synchronous or super-synchronous orbit where synchronous number N, eccentricity e and inclination i are given.

In the above formulae, if the supposition stated before that the sign of $(n+\dot{\omega})$ of retrograde orbits is altered, and if N and periods of retrograde orbits are made negative, the lower sign of double signs must not be used. But because of positiveness of a period, we make N always positive. In the following sections also for the same reason, N is made positive.

The flow chart to calculate a semi-major axis, where the synchronous number N, eccentricity e and inclination i are given, is shown in Fig.2. In this figure, k: geocentric gravitational constant = 398603 km^3/s^2

$$\frac{\partial T}{\partial a} = -2\pi\left(\frac{\partial n}{\partial a} + \frac{\partial \dot{\omega}}{\partial a} \pm \frac{\partial \dot{\Omega}}{\partial a}\right) \Big/ (n + \dot{\omega} \pm \dot{\Omega})^2 \qquad (6)$$

$$\frac{\partial \dot{\Omega}}{\partial a} = \frac{3}{2}\frac{J_2}{P^2}\cos i\left[\left(\frac{2n}{a} - \frac{\partial n}{\partial a}\right)\left[1 + \frac{3}{2}\frac{J_2}{p^2}\left\{\frac{3}{2} + \frac{e^2}{6} - 2\sqrt{1-e^2}\right.\right.\right.$$
$$\left.\left. - \left(\frac{5}{3} - \frac{5}{24}e^2 - 3\sqrt{1-e^2}\right)\sin^2 i\right\}\right] + \frac{3n}{a}\frac{J_2}{p^2}\left\{\frac{3}{2} + \frac{e^2}{6} - 2\sqrt{1-e^2}\right.$$
$$\left.\left. - \left(\frac{5}{3} - \frac{5}{24}e^2 - 3\sqrt{1-e^2}\right)\sin^2 i\right\}\right] + \frac{35}{8}\left(\frac{4J_4}{p^4} + \frac{3}{2}\right)\frac{n_0}{a}\left(1 + \frac{3}{2}e^2\right)$$
$$\cdot\frac{12 - 21\sin^2 i}{14}\cos i$$

The formulae to calculate n, n_0, $\dot{\Omega}$ and $\dot{\omega}$ are given at pp. 214 and the formulae for $\frac{\partial \dot{\Omega}}{\partial a}$ and $\frac{\partial \dot{\omega}}{\partial a}$ are at p.221 of Reference (3).

2-2 In case of the rotating reference plane with the period of the earth's revolution

By the use of the reference plane containing the earth's axis and rotating with the earth's mean revolutional period in the same sense, sub-synchronous, synchronous and super-synchronous orbits are defined as follows:

$$T_e = T_S / N \qquad (7)$$

where T_e : time elapsing between two successive passages of a satellite in the same sense through the reference plane

T_S : mean solar day.

Hereinafter sub-synchronous, synchronous and super-synchronous orbits are to be defined by (7) unless being defined by (1) is mentioned. In the case of this rotating reference plane, the plane NOSAC shown in Fig.1 has angular speed ν on the axis NOS and rotates by νT_e during T_e, where ν is the earth's mean revolutional angular speed. Since the mean angular speed of a satellite projected on the equator is $\pm(n + \dot{\omega}) + \dot{\Omega}$, the mean rotation of the projected satellite is

$$\left\{\pm(n + \dot{\omega}) + \dot{\Omega}\right\} T_e = \pm 2\pi + \nu T_e \qquad (8)$$

or

$$(n + \dot{\omega} \pm \dot{\Omega}) T_e = 2\pi \pm \nu T_e \qquad (8')$$

Substituting $Y_\theta = 2\pi / \nu$ into (14) in p.218 of Reference (3), we have

$$T_S = 2\pi / (\mu - \nu) \qquad (9)$$

Substituting again (8) and (9) into (7), we have

$$\pm(n + \dot{\omega}) + \dot{\Omega} = \pm N\mu + (1 \mp N)\nu \qquad (10)$$

or

$$n + \dot{\omega} \pm \dot{\Omega} = N\mu - (N \mp 1)\nu \qquad (10')$$

Another method of defining sub-synchronous, synchronous and super-synchronous orbits is shown in the following. Defining the orbital period relatively to the terrestial longitude i.e. as the time elapsing between two successive passages of a satellite at the same longitude, a sub-synchronous orbit is the orbit whose period is a sub-multiple of or equal to the mean solar day, a synchronous orbit is the orbit whose period is infinitive, and a super-synchronous orbit is the orbit whose period is the product of $1/(1 \mp N)$ and the mean solar day where N is an integer larger than 2. Namely, a sub-synchronous orbit is defined as follows:

$$T_r = T_S \,/\, (N \mp 1) \tag{11}$$

where T_r : mean time elapsing between two successive passages of a satellite in the same sense through the reference plane of a constant longitude.

Since the earth rotates by μT_r during T_r, the mean revolutional angle of a satellite projected on the equator becomes as follows:

$$\left\{\pm(n + \dot{\omega}) + \dot{\Omega}\right\}\, T_r = \pm 2\pi + \mu T_r \tag{12}$$

Substituting (9) and (12) into (11), we can get (10).
And a super-synchronous orbit is defined as follows:

$$T_r = T_S \,/\, (1 \mp N) \tag{11'}$$

The mean revolutional angle of a satellite projected on the equator during T_r is

$$\left\{\pm(n + \dot{\omega}) + \dot{\Omega}\right\}\, T_r = -2\pi + \mu T_r \tag{12'}$$

Substituting (9) and (12') into (11'), we can also get (10). To define a synchronous orbit, we can use either the set of (11) and (12) or the set of (11') and (12'). The foregoing means that in order to define sub-synchronous, synchronous or super-synchronous orbits, either the plane rotating with the period of the earth's revolution in the same sense or the plane rotating with the angular velocity equal to that of the earth's rotation can be used as the reference plane. We can get the flow chart to calculate a semi-major axis a where synchronous number N, eccentricity e and inclination i are given, substituting T_e for T_f, T_s = 1 day = 86400 sec. for T_θ,

$$T_j = 2\pi \,/\, (n + \dot{\omega} \pm \dot{\Omega} \mp \nu)$$

for the formula to calculate T_j, and the following formula

$$\frac{\partial T}{\partial a} = -2\pi \left(\frac{\partial n}{\partial a} + \frac{\partial \dot{\omega}}{\partial a} \pm \frac{\partial \dot{\Omega}}{\partial a}\right) \Big/ (n + \dot{\omega} \pm \dot{\Omega} \mp \nu)^2$$

for (6), in Fig. 2.

3. Examples of Sub-Synchronous, Synchronous and Super-Synchronous Orbits

Substituting N = 1, which defines synchronous orbits, into (5) and (10), we have

$$\pm (n + \dot{\omega}) + \dot{\Omega} = \pm\mu$$

and $n + \dot{\omega} + \dot{\Omega} = \mu$: direct orbit

$-(n + \dot{\omega}) + \dot{\Omega} = -\mu + 2\nu$: retrograde orbit

The above formulae show that definition of synchronous orbits in Section 2-1 is equivalent to definition in Section 2-2 in case of direct orbits but that they are different in case of retrograde orbits. Although orbits are defined to be synchronous where $N = 1$ according to the term issued by CCIR[(1)], it might be reasonable to define orbits as being synchronous only where $N = 1$ and direct orbits, and as being sub-synchronous where $N = 1$ and retrograde orbits, because in general the term "synchronous" which is used to be contrasted with "sub-synchronous" means to be with the same period and sense of movement.

Substituting the following formula which defines sun-synchronous orbits into (10)[(3)],

$$\dot{\Omega} = \nu$$

we have

$$n + \dot{\omega} = N(\mu - \nu)$$

and substituting again (2) and (9) into the above formula, we have

$$T_n = T_s/N$$

This formula shows that sun-synchronous sub-synchronous orbits equal sun-synchronous recurrent orbits. This is natural because sun-synchronous sub-synchronous satellites pass on nearly the same point at the same time every day since sun-synchronous satellites pass nearly in the same latitude at the same mean solar time every day and sub-synchronous satellites pass nearly in the same longitude at the same standard time every day, and because sun-synchronous recurrent orbits pass on nearly the same point at the same time every day[(3)]. Sub-synchronous, synchronous or super-synchronous satellites pass exactly at the same longitude at the same time every day or at intervals of a few days, only where orbital eccentricity is zero and inclination is zero or π, because in other cases the effects of movement of perigee, equation of centre and reduction to the equator occur to prevent the synchronization. Equation of centre is expressed as follows:

$$\Delta v = 2 \tan^{-1} \sqrt{\frac{1+e}{1-e}} \tan \frac{E}{2} - E + e \sin E$$

and reduction to the equator is as follows:

$$\begin{aligned} \tan(\alpha - \Omega) &= \tan\{\pm(\omega + v)\} \, |\cos i| \\ &= \tan(\omega + v) \cos i \end{aligned} \qquad (13)$$

where E: eccentric anomaly $(0 \leq E < 2\pi)$

v: true anomaly

α: right ascension

and double signs of + and — correspond to direct and retrograde orbits in this order.

Figs. 3—7 show super-synchronous, synchronous and sub-synchronous orbits. Every figure shows the ground track whose ordinate and abscissa represent northern latitude and relative east longitude respectively, the relation of elevation angles and azimuth (except Figs.

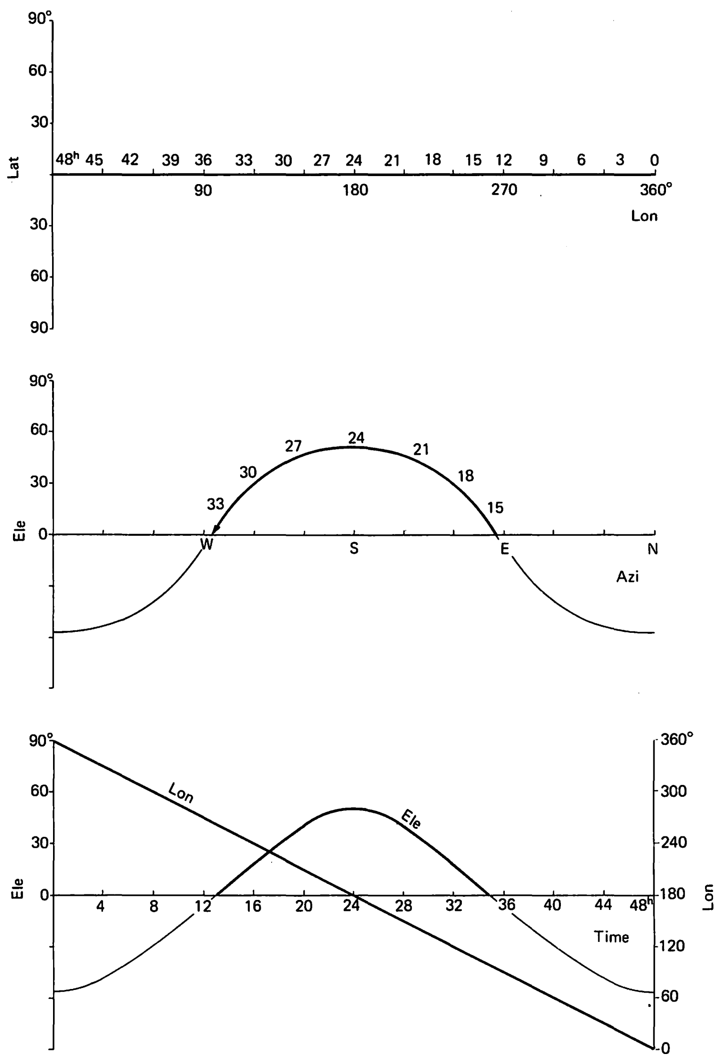

Fig. 3 The ground track, relation of elevation angles and azimuth, and relation of elevation angles, longitude and time, of the super-synchronous orbit with N (: synchronous number) = 0.5, e (: eccentricity), i (: inclination) = 0 and h (: height of the satellite) = 60433 km.

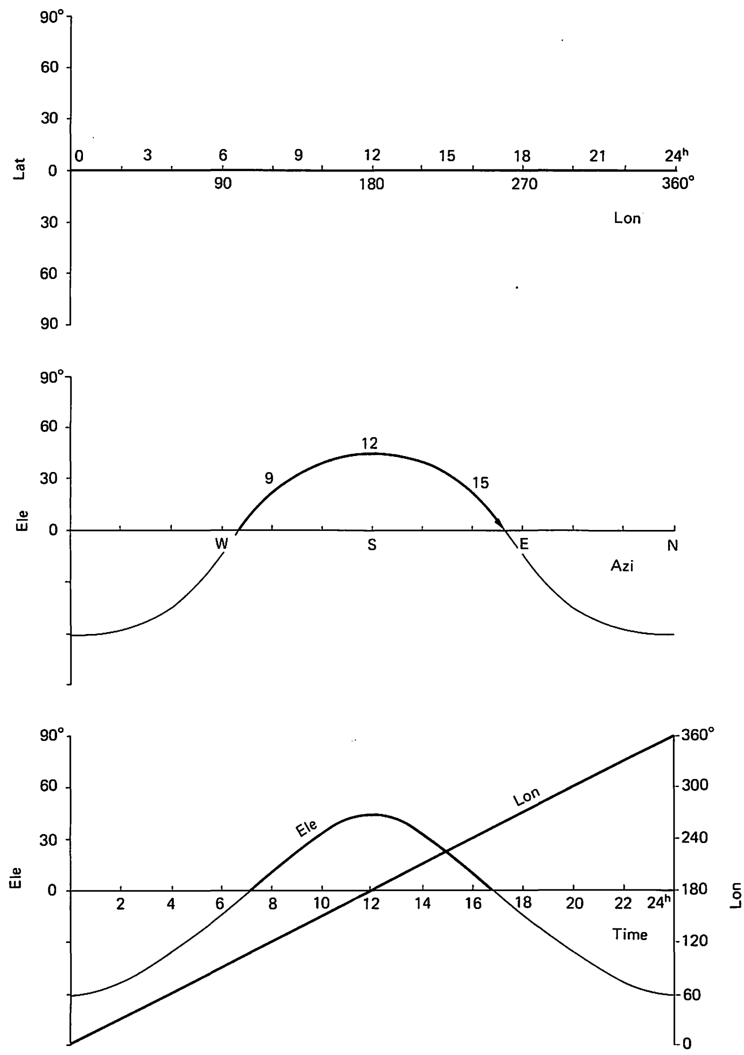

Fig. 4 The ground track, relation of elevation angles and azimuth, and relation of elevation angles, longitude and time, of the sub-synchronous orbit with $N = 2$, $e = 0$, $i = 0$ and $h = 20211$ km.

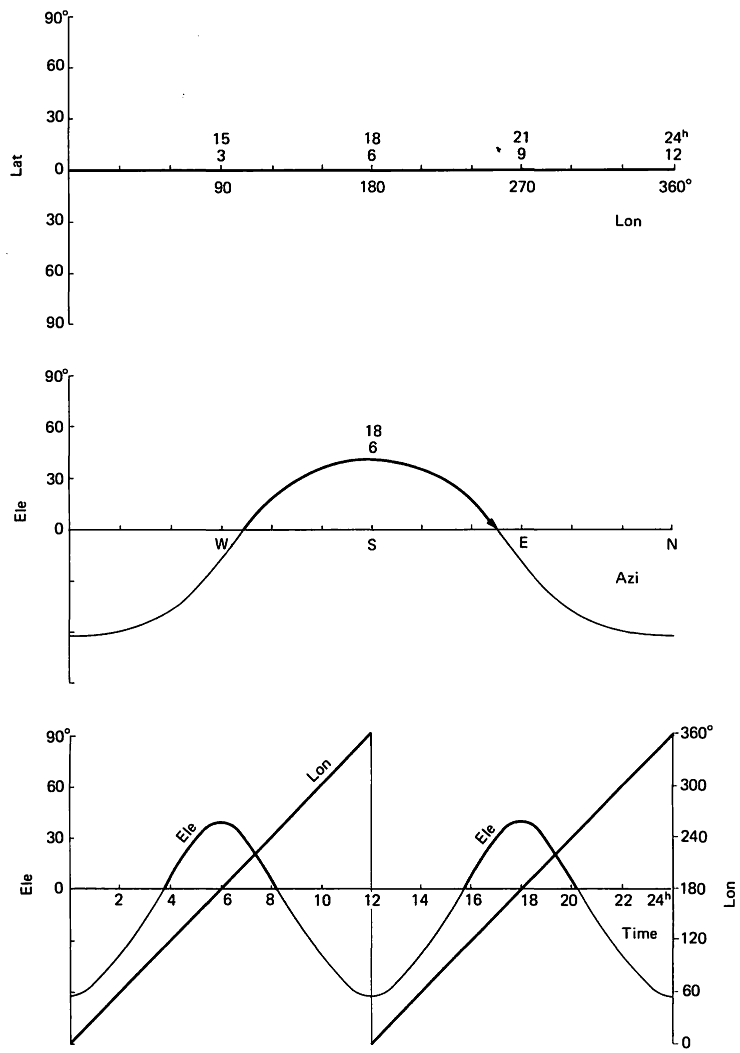

Fig. 5 The ground track, relation of elevation angles and azimuth, and relation of elevation angles, longitude and time, of the sub-synchronous orbit with $N = 3$, $e = 0$, $i = 0$ and $h = 13921$ km.

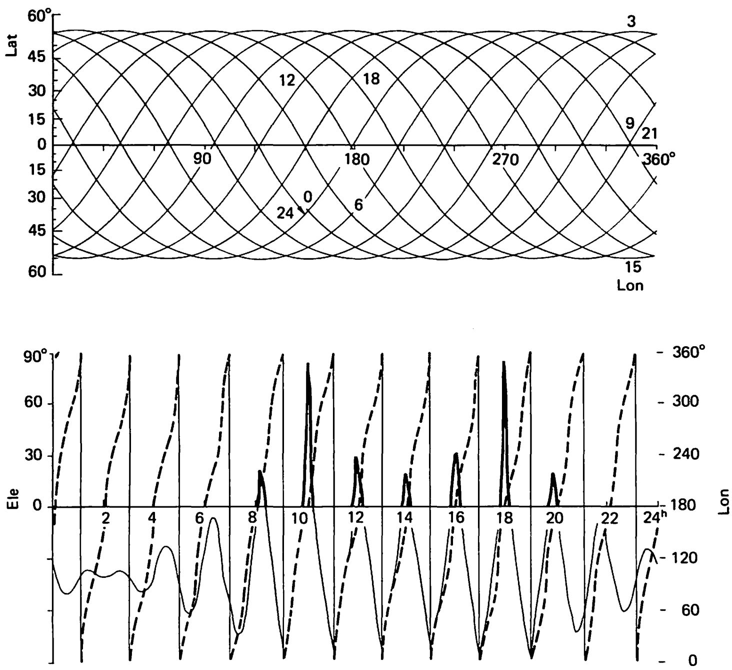

Fig. 6 Mercator's ground track, and relation of elevation angles, longitude and time, of the sub-synchronous orbit with $N = 13$, Epoch = 18^{h}, $\Omega = 29^{\circ}.36$, $\omega = 135^{\circ}.56$, $e = 0$, $i = 55^{\circ}$ and $h = 1260$ km. The solid lines show elevation angles and dashed lines show longitude.

6 & 7), and the relation of elevation angles or longitude and time. The azimuth and elevation in the figures are the values with which the satellite is viewed at 180° E Lon. and 35°.953 N Lat (equal to Kashima Earth Station). Numerals in the figures and the abscissa at the bottom are lapse time in hour from the epoch. Figs. 6 & 7 are examples to show orbits suitable for local geodesic observation. If local geodesic satellies adopt photographing and LASER ranging systems, it will be necessary for the satellite orbits that the height should be about 1000 km, and that the ground track from north to south and the track from south to north should cross each other nearly at right angles. The satellite is usable only when it is night on the ground at the observation point of the satellite and besides, when the satellite is exposed to the sun. Sun-synchronous circular orbits do not satisfy the latter item in the above two conditions, because sun-synchronous circular orbits of 1000 km height have 99°.473 inclination. In case of sub-synchronous orbits shown in Fig. 6 where $N = 13$, the longitude of the satellite at the same time on every day moves a little, and the period of variation of the ground track is about 89 days, and the period as in case of $N = 14$ shown in Fig. 7 is about 78 days. In these figures, ground tracks are drawn on Mercator's projection on which the tracks run at exact angles with each other, to show the tracks to be nearly at right angles

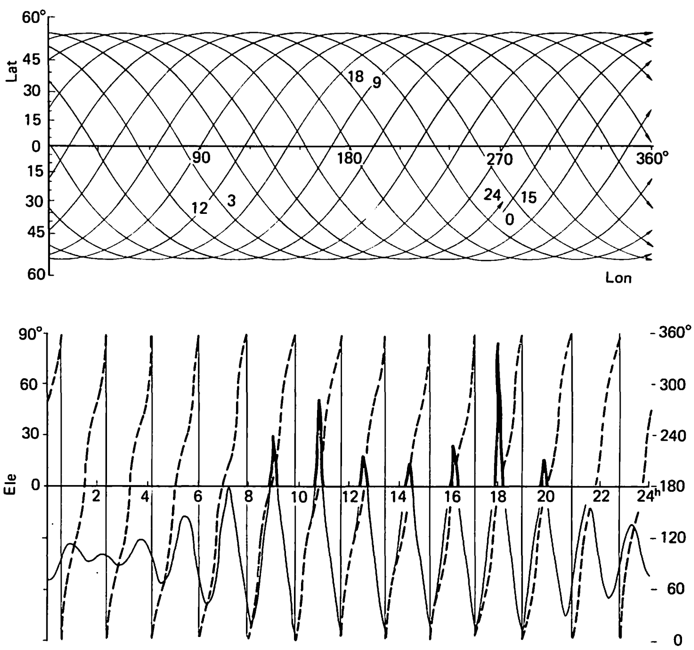

Fig. 7 Mercator's ground track, and relation of elevation angles, longitude and time, of the sub-synchronous orbit with $N = 14$, Epoch = 18^{h}, $\Omega = 29\overset{\circ}{.}36$, $\omega = 135\overset{\circ}{.}56$, $e = 0$, $i = 55^{\circ}$ and $h = 891$ km. The solid lines show elevation angles and dashed lines show longitude.

each other in medium latitudes. If geodesic satellites do not adopt sub-synchronous orbits, it happens that the longitude of the satellite at the same time on every day changes by about 15° at its maximum. Therefore, since chances to use the satellite at a fixed point happen at random, the point to observe the satellite must be moved every day regardless of success in the observation. On the contrary, in case of sub-synchronous satellites, chances to use the satellite at a fixed point occur nearly at the same time every day for a few days. Therefore the observation can be made effectively, as a result of moving the observation point only after the observation has succeeded.

4. Conclusions

The methods of determining sub-synchronous, synchronous and super-synchronous orbits are shown which are sub-synchronous, synchronous and super-synchronous respectively with the earth's rotation in an inertial system, i.e. with a mean sidereal day as stated in Section 2-1, or with the earth' rotation relative to the sun, i.e. with a mean solar day as stated in Section 2-2. It is also proved that sun-synchronous sub-synchronous orbits, being sub-synchronous with a mean solar day, equal sun-synchronous recurrent orbits. The sub-synchronous and super-synchronous orbits will be applicable to broadcasting or communications satellites serving in almost all the world at fixed hours every day or at intervals of a few days, and the sub-synchronous orbits are applicable to geodesic satellites serving in low or medium latitudes. For example, radio broadcasting from a sub-synchronous satellite in a zero-inclination circular orbit will be more efficient and reliable than present oversea short wave broadcasting.

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